Dynamic Programming Lecture #9

Outline:

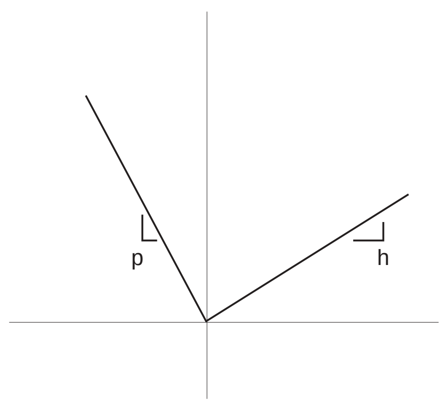
- Inventory control

Inventory Control DP

$$x_{k+1} = x_k + u_k - w_k$$

- $x = \begin{cases} \text{inventory} & x > 0 \\ \text{backlog} & x < 0 \end{cases}$
- $u =$ production
- $w =$ demand
- Define

$$r(z) = \begin{cases} h\,|z| & z > 0 \\ p\,|z| & z < 0 \end{cases}$$



- Total cost:

$$\sum_{k=0}^{N-1} r(x_k + u_k - w_k) + cu_k$$

production cost + sum of either: backlog penalty/holding cost

- Assume: $p > c > 0$ (backlog more costly than production).

DP Algorithm

- General form:

$$J_N(x_N) = g(x_N)$$

$$J_k(x_k) = \min_{u_k} E_{w_k} \{g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k))\}$$

- For inventory control:

$$J_N(x_N) = 0$$

$$\begin{aligned} & J_{N-1}(x_{N-1}) \\ & = \min_{u_{N-1}} E_{w_{N-1}} \{cu_{N-1} + p\max(0, w_{N-1} - x_{N-1} - u_{N-1}) + h\max(0, x_{N-1} + u_{N-1} - w_{N-1}) + J_N(x_N)\} \\ & = \min_{u_{N-1}} (\, cu_{N-1} + E_{w_{N-1}} \{p\max(0, w_{N-1} - x_{N-1} - u_{N-1}) + h\max(0, x_{N-1} + u_{N-1} - w_{N-1})\}\,) \end{aligned}$$

Inventory Control DP, cont (2)

- Define

$$\begin{aligned} H(z) &= E_w r(z-w) \\ &= \sum_{p_i} p_i r(z-w_i) \end{aligned}$$

- For example, if

$$w = \begin{cases} 0 & \text{with probability } p_0 \\ 1 & \text{with probability } 1-p_0 \end{cases}$$

  Then

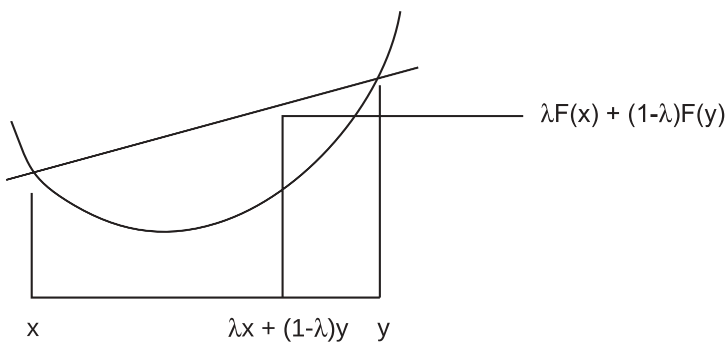
$$H(z) = p_0 r(z) + (1-p_0) r(z-1)$$

- Note that H is a DETERMINISTIC function of z.
- Second look:

$$J_{N-1}(x_{N-1}) = \min_{u_{N-1}\geq 0} \left(c u_{N-1} + H(x_{N-1}+u_{N-1}) \right)$$

- Searching for structure:
  - In LQ optimal control: quadratic $J_{k+1} \Rightarrow$ quadratic J_k.
  - We will pursue similar structure for inventory control.

Background: Convex Functions



- DEFINE: $F : \mathcal{R} \to \mathcal{R}$ convex if for all $x, y \in \mathcal{R}$:

$$F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y), \quad \forall \lambda \in [0,1]$$

- Important implication: If x^* is a local minimum, then x^* is a global minimum.
- PROOF: Suppose $F(y^*) < F(x^*)$ and $y^* > x^*$. Inspect:

$$F(\lambda y^* + (1-\lambda)x^*) \leq \lambda F(y^*) + (1-\lambda)F(x^*)$$
$$F(x^* + \lambda(y^* - x^*)) < F(x^*)$$
$$F(\text{point near } x^*) < F(x^*)?!$$

- Furthermore, if $\lim_{|x|\to\infty} F(x) = \infty$, then F achieves minimum.
- Define DETERMINISTIC function:

$$\tilde{F}(x) = E_w \{F(x+w)\}$$

  If F is convex, so is $\tilde{F}$.

- PROOF: For $\tilde{F}(x) = \sum p_i F(x + w_i)$ and $\tilde{F}(y) = \sum p_i F(y + w_i)$

$$\begin{aligned}
\tilde{F}(\lambda x + (1-\lambda)y) &= \sum p_i F(\lambda x + (1-\lambda)y + w_i) \\
&= \sum p_i F(\lambda(x + w_i) + (1-\lambda)(y + w_i)) \\
&\leq \sum p_i \lambda F(x + w_i) + (1-\lambda)F(y + w_i) \\
&= \lambda E_w \{F(x+w)\} + (1-\lambda)E_w \{F(y+w)\} \\
&= \lambda \tilde{F}(x) + (1-\lambda)\tilde{F}(y)
\end{aligned}$$

Back to Search for Structure

$$H(z) = E_w\{r(z-w)\}$$

- r convex implies H convex.
- Let $u_{N-1} = \tilde{u}_{N-1} - x_{N-1}$. Then

$$\begin{aligned} J_{N-1}(x_{N-1}) &= \min_{u_{N-1}\geq 0} \left(cu_{N-1} + H(x_{N-1}+u_{N-1}) \right) \\ &= \min_{\tilde{u}_{N-1}\geq x_{N-1}} \left(c\tilde{u}_{N-1} + H(\tilde{u}_{N-1}) \right) - cx_{N-1} \end{aligned}$$

  convex minimization!

- (Unconstrained) Minimum of $c\tilde{u} + H(\tilde{u})$ is achieved:
  - If $\tilde{u} \to \infty$

$$c\tilde{u} + H(\tilde{u}) = c\tilde{u} + h\tilde{u} - hE\{w\} \to \infty$$

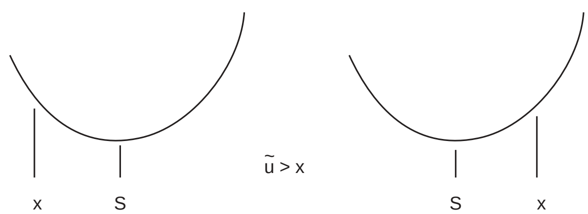
  - If $\tilde{u} \to -\infty$

$$c\tilde{u} + H(\tilde{u}) = (c-p)\tilde{u} + pE\{w\} \to \infty$$

Search for Structure, cont (2)

- Let S_{N-1} be (unconstrained) minimizer of $c\tilde{u}_{N-1} + H(\tilde{u}_{N-1})$

$$\tilde{u}^*_{N-1} = \begin{cases} S_{N-1} & S_{N-1} \geq x_{N-1} \\ x_{N-1} & S_{N-1} < x_{N-1} \end{cases}$$



- Threshold policy:

$$u^*_{N-1} = \begin{cases} S_{N-1} - x_{N-1} & x_{N-1} \leq S_{N-1} \\ 0 & x_{N-1} > S_{N-1} \end{cases}$$

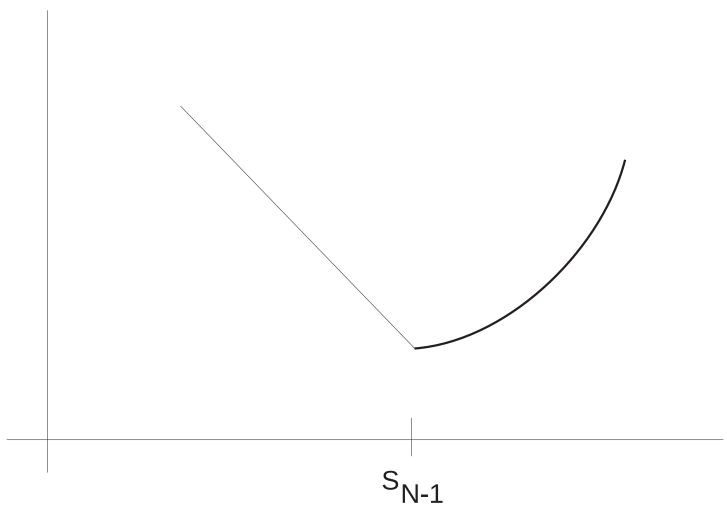
- Interpretation: Bring inventory up to predetermined level S_{N-1}.

Shape of Cost-to-go

- Result:

$$J_{N-1}(x_{N-1}) = \begin{cases} cS_{N-1} + H(S_{N-1}) - cx_{N-1} & x_{N-1} \leq S_{N-1} \\ H(x_{N-1}) & x_{N-1} > S_{N-1} \end{cases}$$

  Convex!



Convex Recursions

- Repeat DP procedure:

$$
\begin{aligned}
&J_{N-2}(x_{N-2}) \\
&\quad = \min_{u_{N-2}} E_{w_{N-2}} \left\{ cu_{N-2} + p\max(0, -x^+) + h\max(0, x^+) + J_{N-1}(x^+) \right\} \\
&\quad = \min_{u_{N-2}} \left(cu_{N-2} + H(x_{N-2} + u_{N-2}) + E_{w_{N-2}} \left\{ J_{N-1}(x_{N-2} + u_{N-2} + w_{N-2}) \right\} \right)
\end{aligned}
$$

- Key features remain same:
  - Above is convex minimization of sum of 3 convex functions.
  - Substitute $u_{N-2} = \tilde{u}_{N-2} - x_{N-2}$.
  - Recognize minimum is achieved.
  - Threshold policy:

$$
u^*_{N-2} = \begin{cases} S_{N-2} - x_{N-2} & x_{N-2} \leq S_{N-2} \\ 0 & x_{N-2} > S_{N-2} \end{cases}
$$

  - Recognize J_{N-2} is convex.
  - Repeat...

Typical DP Features

- Compute a FEEDBACK policy:

$$\mu_k^*(x_k) = \begin{cases} S_k^* - x_k & x_k \leq S_k \\ 0 & x_k > S_k \end{cases}$$

- S_k must be computed in advance.
- Policy CHANGES as horizon approaches.
- DP reveals structure of optimal policy. Can now do guided optimization search over threshold values.
- Structure of optimal policy as/more important than parameters of optimal policy.