A Class of Distributed Adaptive Pricing Mechanisms for Societal Systems with Limited Information

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Abstract—In this paper, we present a class of learning dynamics for distributed adaptive pricing in affine congestion games. We consider the setting where a large population of users is faced with the problem of choosing between a finite number of available resources, each resource having a particular cost function that depends only on the share of users using that particular resource. Since the mass of users is constant, their individual decisions affect the performance of all the available resources, thus generating a population game where each resource can be seen as a particular strategy in the game. Given the well-known fact that Nash equilibria in population games may not be socially optimal, a social planner is faced with the challenge of designing incentive mechanisms that induce a socially optimal Nash equilibrium. To achieve this, we present in this paper a class of model-free distributed pricing algorithms that guarantee convergence to the set of optimal tolls that induce a socially optimal Nash equilibrium. Our results allow us to consider populations of users that react instantaneously to tolls, as well as populations with social dynamics. Since the algorithms are distributed and data-driven, they can be implemented in settings where full information of the game is not available. By combining tools from game theory, robust set-valued dynamical systems, and adaptive control, a convergence result is established.

I. INTRODUCTION

Many of today’s engineered systems are tightly interconnected with their users, and in many cases system performance depends greatly on user behavior [1]. Because of this, game theory is playing a bigger role in the analysis and design of many engineering systems that interact with a large population of users making decisions in real time. These socially integrated engineering systems, also called societal systems, appear in a variety of contexts including transportation networks [2], supply-chain management [3], and electric power grids [4], for example. Since the overall performance of societal systems depends heavily on the behavior of the users, a social planner aims to design methods to influence their behavior in order to induce a positive change on aggregate system performance. One of the most common ways to influence user behavior involves designing tolls or taxes that are priced depending on the particular conditions of the system. For instance, in a traffic routing problem where drivers go from one point to another, a toll may be assigned to each road aiming to minimize the overall delay of the traffic network [1]. This idea can naturally be extended to other applications. Because of this, the problem of designing pricing mechanisms for societal systems has been extensively studied during the last years [5], [2], [6], [7], [1]. For instance, in [5] and [8] the authors presented a class of marginal-based pricing mechanisms corresponding to static mappings that are implemented for a class of social dynamics. In these works, it was shown via Lyapunov arguments that if the population dynamics satisfy a positive correlation property, convergence towards the socially optimal Nash equilibrium is obtained. Flow-varying tolls were also considered in [9] and [10]. However, even though flow-varying tolls based on marginal cost can successfully influence the population towards a socially optimal Nash equilibrium, some limitations prevent their application in practical environments. In particular, as noted in [2], in practical applications users dislike fast-changing tolls, and would prefer prior information of the tolls before making a decision. Moreover, it is well known that fixed tolls can be used to incentivize any feasible Nash equilibria, [11], [12]. One of the challenges in designing real-time pricing mechanisms for tolls comes from the complex nature of large-scale interconnected systems operating under uncertain conditions. Because of this, social planners are also faced with the challenge of characterizing the model of the system by using real-time data. In transportation systems, for example, the delay associated to a road may depend heavily on external factors such as time and weather conditions. Therefore, it is difficult to have a precise characterization of the gradient of the cost function representing the delay, thus further complicating the implementation of marginal tolls. Nevertheless, the delay experienced by drivers that use a particular road can be measured by current technology and infrastructure [2]. This suggests that it is desirable to implement data-driven pricing mechanisms that are agnostic with respect to the exact model of the game, e.g., [6], [10], [13], [14], [15].

Motivated by this, we present in this paper a novel class of distributed adaptive pricing algorithms for dynamic societal systems describing congestion games with affine cost functions and full utilization of resources. The distributed dynamic pricing mechanisms are based on the observation that for this type of games, welfare-gradient dynamics correspond to a type of full-information linear Laplacian flow. The pricing dynamics are game-agnostic, that is, they do not require any knowledge about the mathematical form of the cost functions of the game, and need to measure only the current share of users implementing their individual resource and its cost. Using arguments based on singular perturbations for set-valued dynamical systems we establish a convergence result to a neighborhood of the unknown set of optimal tolls.

The authors are with the Dept. of Electrical and Computer Engineering at the Univ. of California, Santa Barbara. Work supported in part by ONR Grant N00014-17-1-2060 and NSF Grant ECCS-1638214, AFOSR grant number FA 9550-12-1-0127 and NSF grant number ECCS-1232035.
that incentivizes the society in an optimal way.

This paper is organized as follows: Section II describes the setting of pricing in congestion games. Section III presents pricing mechanisms for populations with an Oracle. Section IV shows how to adapt the dynamics for the case when the information of the game is not available. Section V presents some simulations, and finally Section VI ends with some conclusions.

II. CONGESTION GAMES AND THE PROBLEM OF INCENTIVE DESIGN

Consider a population of users with total mass \( m \), where each user can choose between a finite set of available resources \( \mathcal{V} := \{1, \ldots, N\} \), also called strategies. Since the mass of users is constant, we normalize the size of the population to 1, and for each \( i \in \mathcal{V} \) we denote by \( x_i \) the share of users choosing the \( i \)-th resource. We call \( x = [x_1, \ldots, x_N]^\top \) the society state, and we note that \( x \) belongs to the simplex \( \Delta = \{ x \in \mathbb{R}^N_{\geq 0} : 1^\top x = 1 \} \). Each strategy \( i \in \mathcal{V} \) has a related cost given by \( c_i(x_i) \), and we denote the vector of costs of the game by \( c(x) = [c_1(x_1), \ldots, c_N(x_N)]^\top \).

A. Nash flows

When players aim to minimize their individual cost, the steady-state distribution \( x_f \in \Delta \) that emerges in a congestion game is called a Nash flow [16]. Since congestion games are also potential games [8, Sect. 2.4] with potential

\[
P(x) := -\sum_{i=1}^{N} \int_{0}^{x_i} c_i(z) \, dz,
\]

when the costs \( c_i(\cdot) \) satisfy \( \frac{\partial c_i(x_i)}{\partial x_i} > 0 \) for all \( x_i \) and \( i \in \mathcal{V} \), the potential function \( P(\cdot) \) is strictly concave. In this case, a Nash flow \( x_f \) corresponds to the maximizer of \( P(x) \), i.e., the tuple \( (x_f, \mu, \lambda) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \) is the solution of

\[
\begin{align}
\frac{\partial P(x_f)}{\partial x_i} + \lambda_i + \mu &= 0, \quad \forall i \in \mathcal{V} \quad (2a) \\
1^\top x_f &= 1, \quad \lambda_i x_f = 0, \quad \lambda_i \geq 0, \quad \forall i \in \mathcal{V}. \quad (2b)
\end{align}
\]

Since when \( P(\cdot) \) is strictly concave the solution of (2) is unique, Nash flows are also unique. Moreover, since (1) is a separable function, condition (2a) can be rewritten as

\[
-\frac{\partial c_i(x_i)}{\partial x_i} + \lambda_i + \mu = 0, \quad \forall i \in \mathcal{V}. \quad (3)
\]

B. Nash flows and Social Welfare

In a congestion game, the total welfare function \( W(x_f) \) of the system is defined as

\[
W(x_f) := -\sum_{i=1}^{N} c_i(x_{f,i}) x_{f,i} \quad (4)
\]

When (4) is strictly concave the socially optimal flow \( x^* \) corresponds to the flow in the simplex that maximizes \( W(\cdot) \) over \( \Delta \). Thus, \( (x^*, \mu, \lambda) \) satisfies the first order conditions

\[
\begin{align}
\frac{\partial W(x^*)}{\partial x_i} + \lambda_i + \mu &= 0, \quad \forall i \in \mathcal{V} \quad (5a) \\
1^\top x^* &= 1, \quad \lambda_i x^*_i = 0, \quad \lambda_i \geq 0, \quad \forall i \in \mathcal{V}. \quad (5b)
\end{align}
\]

and in this case equation (5a) can be rewritten as

\[
-\frac{\partial c_i(x^*)}{\partial x_i} x^*_i - c_i(x^*_i) + \lambda_i + \mu = 0, \quad \forall i \in \mathcal{V}. \quad (6)
\]

Since the solutions of (2) and (5) are in general not the same, Nash flows \( x_f \) are not necessarily socially optimal.

C. The role of the social planner

Given that Nash flows are in general not socially optimal, a social planner is interested in designing incentive mechanisms for the population, such that the emerging collective behavior is socially optimal. One of the most common ways to incentivize the behavior of the users in a congestion game is to assign tolls \( \tau_i \) to each of the \( N \) resources of the game.

In this way, under the assumption that users are sensitive to the tolls, a new congestion game with tolled costs

\[
c_i(x_i) := c_i(x_i) + s \cdot \tau_i, \quad (7)
\]

is induced by the social planner, where the sensitivity parameter \( s \) characterizes the impact of the toll on the users. Let \( \mathcal{O} : \mathbb{R}^N \to \Delta \) be the function that maps the tolls \( \tau \) to the Nash flow \( x_f \) that solves the linear system (2) with \( c_i(\cdot) \) in (3) replaced by (7), that is, \( \tau \to \mathcal{O}(\tau) \subset \Delta \) being \( \mathcal{S} \subset \Delta \) the set of solutions of (2) with induced cost (7). Note that when \( c_i(\cdot) \) is strictly concave this mapping is well-defined, since each \( \tau \) generates a unique Nash flow. The mapping \( \mathcal{O}(\cdot) \) represents a society that reacts instantaneously to a toll, immediately generating an associated Nash flow \( x_f(\tau) \).

In this case, the challenge for the social planner is to design an adaptive pricing mechanism, i.e., a feedback system, that measures \( x_f(\tau) \), and adjusts the toll \( \tau \) aiming to converge to a Nash flow \( x_f(\tau^*) \) that is also a solution of the global optimization problem (5). Since societal systems are usually large-scale systems, the pricing mechanism associated to the resource \( i \) should share information only with a subset of the pricing mechanisms associated to the other resources.

Because of this, we consider pricing mechanisms with a communication graph \( \mathcal{G} \) satisfying the following assumption.

Assumption 2.1: The graph \( \mathcal{G} \) is weight-balanced and strongly connected.

D. Dynamic Pricing in Fully Utilized Affine Congestion Games

To keep our analysis tractable we consider in this paper a particular class of congestion games characterized by the following assumption.

Assumption 2.2: For the congestion game with \( N \in \mathbb{Z}_{\geq 1} \) resources, the following holds:

- **Affine Costs:** The vector of costs \( c(x) \) is of the form \( c(x) = Ax + b \), where \( b \in \mathbb{R}^N \) and \( A \in \mathbb{R}^{N \times N} \) is a positive definite diagonal matrix.
- **Full utilization:** The mass of users is sufficiently large such that for any \( \tau \in \mathbb{R}^N \) we have that \( \mathcal{O}(\tau) \subset \text{int}(\Delta) \), i.e., Nash flows always satisfy \( x_{f,i} > 0 \) for all resources \( i \in \{1, \ldots, N\} \).

We term congestion games satisfying Assumption 2.2 with arbitrary matrix \( A \) as fully utilized affine congestion games.

Remark 2.1: Note that although the conditions of Assumption 2.2 may seem too restrictive, affine congestion
games emerge in many societal systems such as routing and traffic problems in parallel networks [1], power electrical systems and water distribution systems [4], and resource allocation models in biological and economic systems [17]. Additionally, the full utilization assumption will actually be needed to hold only on compact sets selected a priori by the social planner. In this way, it is natural to expect that when the set of tolls is bounded, and the mass of users is large, there is always at least an \( \epsilon > 0 \) small share of users using every resource \( i \in \mathcal{V} \).

Under Assumption 2.2, the welfare function (4) is strictly concave and has a unique socially optimal flow \( x^* \in \Delta \). However, the set of optimal tolls that generate \( x^* \) may contain more that one element. The following Lemma characterizes this set.

**Lemma 2.1:** Suppose that Assumption 2.2 holds. Then, the set of tolls that generate socially optimal flows via the tolled costs (7), is given by

\[
A_\tau := \{ \tau \in \mathbb{R}^N : \tau = \tau^* + \mu 1, \; \mu \in \mathbb{R} \},
\]

where \( \tau^* = -\frac{b}{2\alpha} \).

**E. The Role of the Social Dynamics**

Even though the mapping \( \mathcal{O}(\cdot) \) described in Section II-C gives a well defined static relation between tolls and Nash flows, in practice, given a new assigned toll to a congestion game, the population of users will require some time to react and adjust their actions accordingly, until convergence to the new induced Nash flow \( x_f(\tau) \) is achieved. This process is modeled by a class of social dynamics called population dynamics [5], [17]. Stability properties of Lipschitz continuous population dynamics have been recently studied in the literature e.g., [18], [5], [4]. To capture this, and more general set-valued social dynamics such as the best-response dynamics, we model the dynamics of the population of users as differential inclusions of the form

\[
\dot{x} \in F(x,u), \quad x \in \Delta, \quad u \in \mathbb{R}^N,
\]

where \( F : \Delta \times \mathbb{R}^N \supseteq T\Delta \) is a set-valued mapping assumed to be outer semicontinuous, locally bounded and convex valued, and which already encodes the vector of cost functions \( c(\cdot) \) of the game. The input \( u \) is the effective vector of tolls perceived by the society, i.e., \( u = s \cdot \tau \), and the output mapping \( h_x(\cdot) \) is the available information for the pricing mechanisms. In this paper we consider population dynamics that satisfy the following two axioms.

**Assumption 2.3:** For all \( u \in \mathbb{R}^N \) the differential inclusion (9a) renders the set \( \Delta \) positive invariant.

**Assumption 2.4:** Consider the social dynamics (9) and a congestion game satisfying Assumption 2.2. For each fixed compact set \( K \subset \mathbb{R}^N \), the \( K \)-constrained system

\[
\dot{x} \in F(x,u) \quad \dot{u} = 0 \quad \text{s.t.} \quad (x,u) \in \Delta \times K,
\]

renders the set \( M_K := \{ (x,u) : x \in \mathcal{O}(u), u \in K \} \) GAS\(^1\).

**Remark 2.2:** In words, Assumption 2.4 asks that for each fixed incentive \( \tau \), all solutions of (9) converge to the unique Nash flow \( x_f(\tau) \) that solves (2).

### III. DYNAMIC PRICING MECHANISMS WITH A SOCIAL ORACLE

In this section we assume that each pricing mechanism \( i \in \mathcal{V} \) has access to the tuple \( \{ x_i, c(x_i), \frac{\partial c(x_i)}{\partial x_i} \} \). Based on this, the output mapping (9b) is defined as

\[
h_x(x_f) := \begin{pmatrix} h_{x,1}(x_f) \\ h_{x,2}(x_f) \\ h_{x,3}(x_f) \end{pmatrix} = \begin{pmatrix} x \\ c(x) \\ \nabla c(x) \end{pmatrix},
\]

where \( \nabla c(x) \) is the Jacobian matrix of \( c(\cdot) \) evaluated at \( x \). We called the output mapping (11) an Oracle, since it assumes full access to \( \nabla c(x) \).

For congestion games where an Oracle is available, we propose the distributed pricing dynamics

\[
\dot{\tau} = \alpha \cdot L_{\mathcal{G}} \left( h_{x,2}(x) + h_{x,3}(x) \cdot h_{x,1}(x) \right),
\]

where \( L_{\mathcal{G}} \) is the Laplacian matrix associated to the communication graph \( \mathcal{G} \) described by Assumption 2.1, and \( \alpha \in \mathbb{R} > 0 \) is a tunable gain. For reasons that will become evident later, we called this dynamics the distributed welfare-gradient dynamics. The following theorem corresponds to the first main result of this paper. Without loss of generality we state the result for the case when \( s = 1 \) (7), since otherwise one can simply define the output of (12) as \( \frac{\dot{\tau}}{s} \).

**Theorem 3.1:** Suppose that Assumptions 2.1-2.4 hold. Then for each triple \( (\rho_1, \rho_2, \epsilon) \in \mathbb{R}^2_0 \) there exists \( \alpha^* \in \mathbb{R}_0 > 0 \) such that for each \( \alpha \in (0, \alpha^*] \) every solution \( (x, \tau) \) of the interconnection of the learning dynamics (12) and the social dynamics (9) with \( u = \tau \) and initial conditions satisfying \( x(0) \in \Delta \) and \( \tau(0) \in \{ \tau(0) : \|\tau\|_{A_\tau} \leq \rho_1, \left( \tau(0) - \tau^* \right) \leq \rho_2 \} \), converges in finite time to an \( \epsilon \)-neighborhood of the point \( \{ x^* \} \times \{ \tau^*_0 \} \in \Delta \times A_\tau \) where

\[
\tau^*_0 := \tau^* + \frac{1}{N} \left( \tau(0) - \tau^* \right),
\]

and where \( \tau^* = -\frac{b}{2\alpha} \) and \( x^* \) is the socially optimal flow.

The convergence result of Theorem 3.1 is based on inducing a time-scale separation between the social dynamics (9) and the pricing dynamics (12), such that tolls vary sufficiently slowly in comparison to the dynamics of the users. Because of this, users have enough time to rationally react to variations of tolls.

**Remark 3.1:** The constants \( \rho_1 \) and \( \rho_2 \) in Theorem 3.1 are simply used to characterize a compact set of initial conditions for the tolls. Since they can be selected arbitrarily, the result from Theorem 3.1 is of semi-global practical nature.

We specialize the results of Theorem 3.1 for two particular cases of interest: 1) Pricing mechanisms that assume “instantaneous” population dynamics, and 2) Pricing mechanisms with an undirected complete communication graph.

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\(^1\)We use the acronym “GAS” for Globally Asymptotically Stable, [19, Section 2]
Corollary 3.2: Suppose that Assumptions 2.1 and 2.2 hold. Consider the distributed pricing dynamics (12) with \( x \) replaced by its steady-state mapping \( x_f(\tau) = O(\tau) \). Then every solution of (12) asymptotically converges to the point \( \tau_0^* \) in (13). Moreover, if the adjacency matrix of the graph \( \mathcal{G} \) is given by \( \text{Adj} = A^{-1}11^T A^{-1} \), then the pricing mechanism (12) is equivalent to the welfare-gradient dynamics

\[
\dot{\tau} = \alpha \cdot \frac{1}{k} \nabla_r \tilde{W}(\tau),
\]

(14)

where \( \tilde{W} := W \circ O(\tau) \) and \( k = (1^T A^{-1}1)^{-1} \in \mathbb{R} \). □

The second result of Corollary 3.2 reveals that the pricing dynamics (12) can be seen as a distributed version of the full-information welfare-gradient dynamics (14), which motivates us to employ the term distributed welfare-gradient dynamics.

By identifying the welfare-gradient dynamics as a type of Laplacian flow of the form (12), one can exploit existing results in consensus dynamics and saddle point dynamics to additionally optimize a particular performance index \( J(\cdot) \) defined over the set of optimal tolls \( A \). Examples of common performance indexes include \( J(\tau) = \tau^T x_f(\tau) \), and \( J(\tau) = ||\tau||_p^p \), with \( p \geq 1 \), which correspond to the total revenue of the pricing mechanism, and the \( p \)-norm of the vector of tolls, respectively. For instance, when \( p = 2 \) the pricing mechanisms should converge to the toll \( \tau^* \) that additionally solves the problem

\[
\min_{\tau \in A} J(\tau) = ||\tau||_2^2 \quad \text{s.t.} \quad \tau \in A,
\]

(15)

which, using the definition of \( A \), is equivalent to

\[
\min_{\mu \in \mathbb{R}} J(\mu) = ||\tau^* + \mu 1||_2^2 = \sum_{i=1}^N J_i(\mu),
\]

(16)

where \( J_i(\mu) = (\mu + \tau_i^*)^2 \), and \( \tau^* = \frac{-b}{2} \). It is easy to see that the optimal solution \( \mu^* \) of (16) satisfies

\[
\mu^* = \frac{1^T b}{2N} \quad \text{for} \quad (16) \quad \Rightarrow \quad \tau^* = -\frac{b}{2} + \frac{1^T b}{2N} 1 \quad \text{for} \quad (15).
\]

To converge to the solution \( \tau^* \) of (15) consider the saddle-point-like dynamics

\[
\begin{align*}
\dot{\tau} &= \mathcal{L} \left( h_{x,2}(x) + h_{x,3}(x) \cdot h_{x,1}(x) - 2z \right) - 2 \cdot \nabla J(\tau) \\
\dot{z} &= \mathcal{L} \left( h_{x,2}(x) + h_{x,3}(x) \cdot h_{x,1}(x) \right).
\end{align*}
\]

(17)

The following proposition characterizes the convergence of (17). Due to space limitations we specialize the result for the case when the social dynamics are neglected.

Proposition 3.3: Suppose that Assumptions 2.1 and 2.2 hold, and consider the dynamics (17) with \( x \) replaced by the steady-state Nash flow \( x_f(\tau) \). Then, each solution of (17) remains bounded and \( \tau \) converges asymptotically to \( \tau^* \). □

IV. ADAPTIVE PRICING MECHANISMS WITH POPULATION EXCITATION

The results of the previous section were based on the existence of an Oracle that delivers samples of the gradient of the costs \( c_i(\cdot) \). However, in many cases this assumption is unrealistic and difficult to satisfy. In this section we address this issue by making the observation that even though the results of Theorem 3.1 can be seen as a robustness result for the distributed welfare dynamics (12) under fast social dynamics of the form (9), they also open the door for the exploitation of the social dynamics (9) to learn information from the underlying game. To achieve this, let \( \tilde{c}_i(x) = \tilde{w}_i^T \phi_i(x_i) \) be an approximation of the cost function \( c_i(\cdot) \), where \( \tilde{w}_i \in \mathbb{R}^m \) is a vector of tunable weights, and \( \phi_i : \mathbb{R} \to \mathbb{R}^m \) is a continuous vector-valued basis function. Since the parameters of the game are \( \tilde{w}_i^* = [A_{i,i}, b_i]^T \), a simple choice of basis functions for affine congestion games is given by \( \phi_i = [x_i, 1]^T \), for all \( i \in \mathcal{V} \). To learn these parameters online consider the learning dynamics

\[
\hat{w}_i = -\frac{\phi_i(x_i)}{(1 + \phi_i(x_i)^T \phi_i(x_i))} \left( \hat{c}_i(x_i) - c_i(x_i) \right),
\]

(18)

which can be seen as a normalized gradient-based algorithm of the estimation error [20, Sec. 4.3.5], [21]. System (18) needs only to measure the value of the social state \( x_i \) and cost function \( c_i(x_i) \) associated to the \( i \)-th resource. Therefore, the output (9b) of the social dynamics is now defined as

\[
h_x(x) = (h_{x,1}(x), h_{x,2}(x))^T = (x, c(x))^T.
\]

(19)

To learn the cost parameters of the resource \( i \), the \( i \)-th pricing mechanism must provide enough excitation to the population. This is achieved by using the signal generator

\[
\mu_i \in \Psi_i(\mu_i), \quad \mu_i \in \Lambda_i,
\]

(20a)

\[
y_i = h_{\mu_i}(\mu_i),
\]

(20b)

where the set \( \Lambda_i \subset \mathbb{R}^\ell \) is compact, \( \ell \in \mathbb{Z}_{\geq 1} \), the set-valued mapping \( \Psi_i : \mathbb{R}^\ell \Rightarrow \mathbb{R}^\ell \) is outer-semicontinuous, locally bounded, and convex-valued, and \( h_{\mu_i} : \mathbb{R}^\ell \to \mathbb{R} \) is continuous. The signal generator is assumed to satisfy the following stability assumption.

Assumption 4.1: The dynamic signal generator (20) renders a nonempty compact set \( \mathcal{A}_{\mu_i} \subset \Lambda_i \) GAS.

Assumption 4.1 is mainly needed to guarantee a uniform bound property on the signals \( \mu_i \). Standard signal generators of the form (20) include linear oscillators defined on the circle \( \Lambda_i = \mathbb{S} \), or nonlinear systems rendering a limit cycle GAS.

Using the excitation signal \( \mu_i \) and the output function (20b), the overall toll charged to the users is now

\[
\tau_{\text{total}} = \tau + \text{diag}(a) \cdot h_{\mu_i}(\mu_i),
\]

(21)

where \( \mu := [\mu_1, \ldots, \mu_N]^T \), \( \text{diag}(a) \) is a \( N \times N \) diagonal matrix with diagonal entries given by the vector \( a = [a_1, \ldots, a_N]^T \), with \( a_i \in \mathbb{R}_{>0} \) for all \( i \in \mathcal{V} \), and \( h_{\mu_i}(\mu_i) := h_{\mu_i,1}(\mu_i) \times \cdots \times h_{\mu_i,N}(\mu_i) \). The nominal toll \( \tau \) is generated by the distributed welfare-gradient dynamics

\[
\dot{\tau} = \alpha \cdot \mathcal{L} \left( h_{x,2}(x) + \tilde{w}^T \nabla \phi(h_{x,1}(x)) \cdot h_{x,1} \right),
\]

(22)

where \( \alpha \in \mathbb{R}_{>0} \), and where \( \tilde{w}^T \nabla \phi(h_{x,1}) \) is a diagonal matrix with entries given by the vector \( \left[ \frac{\partial \phi_i(x_i)}{\partial x_1}, \ldots, \frac{\partial \phi_N(x_N)}{\partial x_N} \right] \). To characterize the convergence properties of the closed-loop system we first review the notion of persistently exciting signals.
Convergence to Socially Optimal Nash Flow

Dynamics

Nash Flow

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such that welfare-gradient dynamics (22) and the learning dynamics (18) allows to dispense with the assumption on the existence of an Oracle. This excitation property is formalized by the following assumption.

Assumption 4.2: Let $K \subset \mathbb{R}^N$ be a compact set, and consider the system

\begin{equation}
\begin{aligned}
\dot{\varphi} &= \Psi(\varphi)
\end{aligned}
\end{equation}

where $\Lambda := \Lambda_1 \times \ldots \times \Lambda_N$, $\Psi(\varphi) := \Psi_1(\mu_1) \times \ldots \times \Psi_N(\mu_N)$. There exists $a^* \in (0, a)$ such that for all amplitude vectors $a \in (0, a)$ there exists $a^* \in (0, a)$ such that for all $a \in (0, a)$ every complete solution $(\mu, \tau, \dot{x})$ of the closed-loop system with $\mu(0) \in \Lambda, x(0) \in \Delta, \|\dot{w}(0) - w^*\| \leq \rho$ and $\tau(0) \in \{\tau : \|\tau\|_\infty \leq \beta_1, 1^T (\tau - \tau^*) = \beta_2\}$ converges to an $\varepsilon$-neighborhood of the compact set

\begin{equation}
A_{\text{closed-loop}} = A_\mu \times \{\tau_0\} \times \{x^*\} \times \{w^*\},
\end{equation}

where $\tau_0$ is given by (13) and $x^*$ is socially optimal. □

In words Theorem 4.1 says for each compact set of initial conditions, and for any arbitrarily small $\varepsilon$-neighborhood of the socially optimal state $x^*$, it is always possible to select the amplitude of the excitation signals and the gain of the distributed welfare gradient dynamics, such that the adaptive pricing mechanism, being agnostic to the game, incentivizes the society state $x$ towards an $\varepsilon$-neighborhood of the socially optimal state $x^*$. Figure 1 summarizes the results of this paper.

V. NUMERICAL EXAMPLE

To illustrate the geometrical interpretation of the pricing mechanism, we present in this section a simple example of an affine congestion game with two resources, i.e., $\mathcal{V} = \{1, 2\}$, and matrices $A$ and $b$ with entries $A_{1,1} = 3, A_{2,2} = 2, b_1 = 2.5, b_2 = 2$. First, we consider the case when the pricing mechanisms have access to an Oracle that delivers to each mechanism $i \in \mathcal{V}$ sampled values of $c_i(x_i)$ and $\frac{\partial c_i(x_i)}{\partial x_i}$. For the social dynamics we consider the Brown-von Neumann-Nash dynamics [5], i.e.,

\begin{equation}
\dot{x}_i = \max\{0, \bar{c}_i(x, \tau)\} - x_i \sum_{j \in \mathcal{V}} \max\{0, \bar{c}_j(x, \tau)\},
\end{equation}

where $\bar{c}_i(x, \tau) = \hat{c}_i(x_i, \tau_i) - \sum_{j \in \mathcal{V}} \hat{c}_j(x_j, \tau_j) x_j$, and $\hat{c}_i(x_i, \tau_i) = -(A_{i,i} x_i + b_i + \tau_i)$, for all $i \in \mathcal{V}$, which satisfy Assumptions 2.3 and 2.4. Figure 2 shows the convergence of the tolls to the unknown set of optimal tolls $\mathcal{A}$ that incentivizes socially optimal states, from four different initial conditions. The red points denote the theoretically optimal toll $\tau^*_0$ given by (13). The evolution in time of the toll $\tau$ for the case when $\tau(0) = [-2, -2.5]^T$ is shown in Figure 3. The inset shows the evolution of the social state $x$ towards the socially optimal Nash equilibrium. To address the case when the parameters of the game are unknown for the pricing mechanisms, we also simulate the adaptive pricing mechanisms with population excitation presented in Section IV. In this case, for each pricing mechanism we use a signal generator (20) with dynamics $\mu_1 = -\omega_1 \mu_2, \mu_2 = \omega_1 \mu_1$, constrained to the unit circle $S$, and state $\mu_{3,i} = -\beta_i (\mu_{3,i} + \epsilon)$, where $\beta_i \in \mathbb{R}_{>0}$, and output $y_i = h_{\mu,i}(\mu_i) = \mu_{3,i} \mu_{1,i}$. Which generates an output that is an exponentially decaying sinusoidal signal. This system render the set $S \times \{\epsilon\}$ UGAS, thus satisfying Assumption 4.1. The frequencies of the oscillator are selected as $\omega_1 = 40$ and $\omega_2 = 50$. Figure 4 shows the
evolution in time of the pricing tolls $\tau_i$ converging to its optimal value on the set $A$. The convergence to the optimal toll is achieved in approximately 6 seconds. Figure 5 shows the evolution in time of the social state, which slowly decays to a neighborhood of the optimal social state. In practice the excitation signals $h_{\mu,i}$ could be turned off after an initial learning phase, which in this case takes approximately 10 seconds. After turning off the excitation signal the social state $x$ settles to a value $\varepsilon$-close to $x^*$. 

VI. CONCLUSIONS

This paper presents novel distributed adaptive pricing mechanisms for a class of societal systems modeled as affine congestion games with set-valued social dynamics. The learning dynamics can be seen as a type of distributed welfare-gradient dynamics. By using tools from game theory, robust set-valued dynamical systems, and adaptive control, we characterize the stability and convergence properties of these dynamics for the case when they have access to an Oracle, as well as for the case when the Oracle does not exist and the parameters of the game must be learned online by persistently exciting the population via the tolls.

REFERENCES


VII. APPENDIX

Due to space limitations, the complete proofs of this paper can be found online at the url (copy and paste the link):

https://docs.google.com/viewer?a=v&pid=sites&srcid=ZGVmYXVsdGRvbWFpbnxqb3JnZGF8Z3g6MTM0OTZkMGQ4OGJmZjA3MA
12YW5wbzZ1ZGF8Z3g6MTM0OTZkMGQ4OGJmZjA3MA