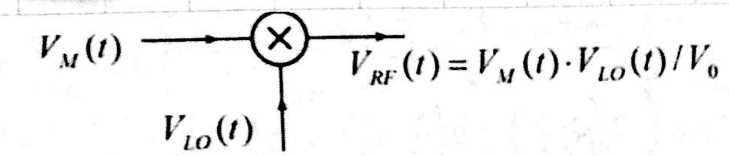
Solutions

Problem 1: The local oscillator is a 1.6MHz cosine wave of 1 V amplitude. The message $V_M(t)$ is a 100 mV *sine* wave at 1kHz. V_0 is 0.1 Volts



(a) Compute $V_{RF}(t)$. (b) Make a clean graph of this using your favorite computer software. (c) Compute the Fourier Transform of $V_{RF}(t)$. Note carefully the phases of the sidebands. (d) If $V_{RF}(t)$ is delivered to a 50 Ohm load resistor, compute the *power* at each frequency in its Fourier spectrum.

a. $V_M(t) = 0.1\sin(2000\pi t) = \frac{1}{20j}e^{j\omega_M t} - \frac{1}{20j}e^{-j\omega_M t} = \frac{1}{20j}Z_M^{1} - \frac{1}{20j}Z_M^{-1}$

$V_{LO}(t) = \cos(3.2\times10^6\pi t) = \frac{1}{2}e^{j\omega_{LO}t} + \frac{1}{2}e^{-j\omega_{LO}t} = \frac{1}{2}Z_{LO}^{1} + \frac{1}{2}Z_{LO}^{-1}$

$$\longrightarrow \left(\frac{1}{20j}\right)\left(\frac{1}{2}\right)\left(Z_M^1 - Z_M^{-1}\right)\left(Z_{LO}^1 + Z_{LO}^{-1}\right) = \left(\frac{1}{40j}\right)\left(Z_M^1Z_{LO}^1 + Z_M^1Z_{LO}^{-1} - Z_M^{-1}Z_{LO}^{1} - Z_M^{-1}Z_{LO}^{-1}\right)$$

$$= \left(\frac{1}{40j}\right)\left(\left(e^{j\omega_M t}e^{j\omega_{LO}t} - e^{-j\omega_M t}e^{-j\omega_{LO}t}\right) + \left(e^{j\omega_M t}e^{-j\omega_{LO}t} - e^{-j\omega_M t}e^{j\omega_{LO}t}\right)\right)$$

$$= \frac{1}{40j}\left(2j\left(\sin((\omega_M+\omega_{LO})t) + \sin((\omega_M-\omega_{LO})t)\right)\right)$$

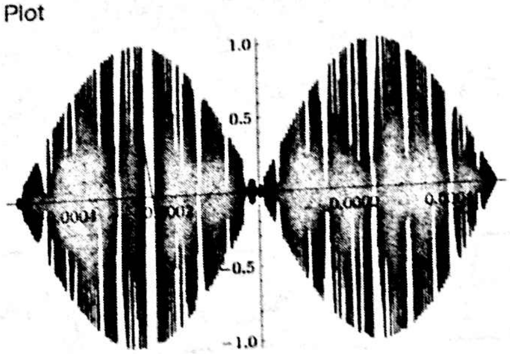
$$= \frac{1}{20}\sin((\omega_M+\omega_{LO})t) + \frac{1}{20}\sin((\omega_M-\omega_{LO})t) = .1\sin(\omega_M t)\cos(\omega_{LO}t)$$

$$V_{RF}(t) = \frac{0.1\sin(\omega_M t)\cos(\omega_{LO}t)}{0.1} = \boxed{\frac{1}{2}\sin(3.202\times10^6\pi\cdot t) - \frac{1}{2}\sin(3.198\times10^6\pi\cdot t)}$$

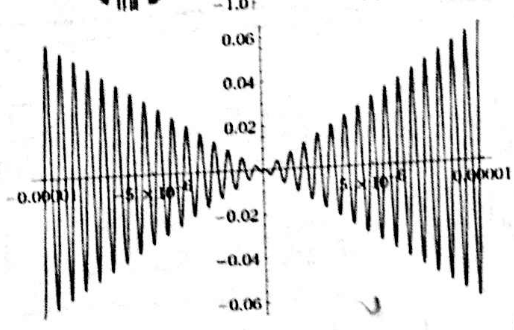
b.

plot $0.5(-\sin(1.00468\times10^7 t) + \sin(1.00594\times10^7 t))$

Plot



$t = -\frac{1}{2000}$ to $\frac{1}{2000}$



$t = -\frac{1}{100000}$ to $\frac{1}{100000}$

c. $V_{RF}(t) = \frac{1}{2}\sin(3.202\times10^6\pi\cdot t) - \frac{1}{2}\sin(3.198\times10^6\pi\cdot t)$

$F\{V_{RF}(t)\} = \frac{1}{2}\left(F\{\sin(3.202\times10^6\pi\cdot t)\} - F\{\sin(3.198\times10^6\pi\cdot t)\}\right)$

$= \frac{1}{2}\left(F\left\{\frac{e^{j\omega_1 t}-e^{-j\omega_1 t}}{j2}\right\} - F\left\{\frac{e^{j\omega_2 t}-e^{-j\omega_2 t}}{j2}\right\}\right) = \frac{1}{j4}\left(F\{e^{j\omega_1 t}\} - F\{e^{-j\omega_1 t}\} - F\{e^{j\omega_2 t}\} + F\{e^{-j\omega_2 t}\}\right)$

$= \frac{1}{j4}\left[2\pi\left(\delta(\omega-\omega_1) - \delta(\omega+\omega_1) - \delta(\omega-\omega_2) + \delta(\omega+\omega_2)\right)\right]$

$$\boxed{= -j\frac{\pi}{2}\left[\delta(\omega-3.202\times10^6\pi) - \delta(\omega+3.202\times10^6\pi) - \delta(\omega-3.198\times10^6\pi) + \delta(\omega+3.198\times10^6\pi)\right]}$$

d. $P = \frac{V^2}{R}$,

$P_1 = \left[-j\frac{\pi}{2}\delta(\omega-3.202\times10^6\pi)\right]^2\cdot\frac{1}{50\Omega} = -\frac{\pi^2}{200}\delta(\omega-3.202\times10^6\pi)^2$

$P_2 = \left[+j\frac{\pi}{2}\delta(\omega+3.202\times10^6\pi)\right]^2\cdot\frac{1}{50\Omega} = -\frac{\pi^2}{200}\delta(\omega+3.202\times10^6\pi)^2$

$P_3 = \left[+j\frac{\pi}{2}\delta(\omega-3.198\times10^6\pi)\right]^2\cdot\frac{1}{50\Omega} = -\frac{\pi^2}{200}\delta(\omega-3.198\times10^6\pi)^2$

$P_4 = \left[-j\frac{\pi}{2}\delta(\omega+3.198\times10^6\pi)\right)\cdot\frac{1}{50\Omega} = -\frac{\pi^2}{200}\delta(\omega+3.198\times10^6\pi)^2$

Problem 2: With the same parameters as problem 1, now $V_M(t)$ is a 100 mV *cosine* wave at 1kHz. (a) Again compute the Fourier Transform of $V_{RF}(t)$. Please explain how the sidebands differ in the cases of problem #1 and problem #2.

a. $V_M(t) = 0.1\cos(2000\pi t)$ $\qquad V_{LO}(t) = \cos(3.2\times10^6\pi t)$

$\frac{1}{20}e^{j\omega_M t} + \frac{1}{20}e^{-j\omega_M t}$ $\qquad \frac{1}{2}e^{j\omega_{LO}t} + \frac{1}{2}e^{-j\omega_{LO}t}$

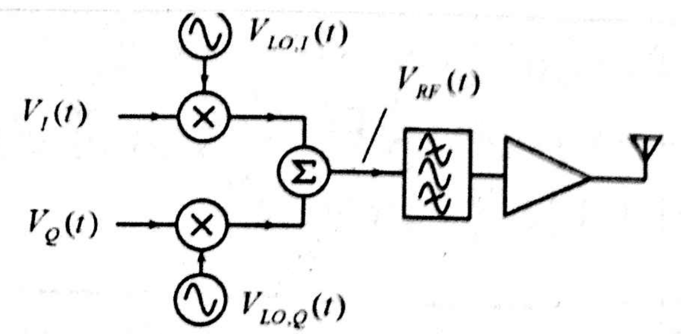
$\left(\frac{1}{20}\right)\left(\frac{1}{2}\right)\left(e^{j\omega_M t} + e^{-j\omega_M t}\right)\left(e^{j\omega_{LO}t} + e^{-j\omega_{LO}t}\right) = \left(\frac{1}{40}\right)\left(e^{j(\omega_M+\omega_{LO})t} + e^{-j(\omega_M+\omega_{LO})t} + e^{j(\omega_M-\omega_{LO})t} + e^{-j(\omega_M-\omega_{LO})t}\right)$

$= \frac{1}{40}\left(2\cos((\omega_M+\omega_{LO})t) + 2\cos((\omega_M-\omega_{LO})t)\right) = V_M(t)\cdot V_{LO}(t)$

$V_{RF}(t) = V_M(t)\cdot V_{LO}(t)/V_0 = \boxed{\frac{1}{2}\cos((\omega_M+\omega_{LO})t) + \frac{1}{2}\cos((\omega_M-\omega_{LO})t)}$

The phase of the sidebands in problem #1 are ±90° whereas the phase in problem #2 is 0° or 180°

Problem 3: This is called Quadrature amplitude modulation. QAM is widely used in modern digital wireless links. We will work a simplified case. $V_{LO,I}(t)$ is a 2.4 GHz cosine wave of 1 V amplitude. $V_{LO,Q}(t)$ is a 2.4 GHz *sine* wave of *-1* V amplitude.



V_o is 0.1 Volts. $V_I(t)$ is the sum of a 100 mV *sine* wave at 1kHz and a 100 mV *cosine* wave at 2 kHz. $V_Q(t)$ is the sum of a 100 mV *sine* wave at 3kHz and a 100 mV *cosine* wave at 4 kHz. (a) Compute $V_{RF}(t)$ (b) Compute the Fourier Transform of $V_{RF}(t)$. Note carefully the phases of the sidebands. (c) Try to explain in words how a receiver might be able to determine $V_I(t)$ and $V_Q(t)$ from the received signal $V_{RF}(t)$.

a. $V_{LO,I}(t) = \cos(4.8\times10^9\pi t)$, $V_{LO,Q}(t) = -\sin(4.8\times10^9\pi t)$,

$V_I(t) = 0.1\sin(2000\pi t) + 0.1\cos(4000\pi t)$, $V_Q(t) = 0.1\sin(6000\pi t) + 0.1\cos(8000\pi t)$

$$V_{RF}(t) = \left(V_I(t)\cdot V_{LO,I}(t) + V_Q(t)\cdot V_{LO,Q}(t)\right)/V_o$$

$$= \Big(\underbrace{\left[\tfrac{1}{20j}Z_I^{1} - \tfrac{1}{20j}Z_I^{-1} + \tfrac{1}{20}Z_{I,2}^{1} + \tfrac{1}{20}Z_{I,2}^{-1}\right]\left[\tfrac{1}{2}Z_{LO,I}^{1} + \tfrac{1}{2}Z_{LO,I}^{-1}\right]}_{X_1} + \underbrace{\left[\tfrac{1}{20j}Z_Q^{1} - \tfrac{1}{20j}Z_Q^{-1} + \tfrac{1}{20}Z_{Q,2}^{1} + \tfrac{1}{20}Z_{Q,2}^{-1}\right]\left[-\tfrac{1}{20j}Z_{LO,Q}^{1} + \tfrac{1}{20j}Z_{LO,Q}^{-1}\right]}_{X_2}\Big)$$

$X_1 \rightarrow$

$$\tfrac{1}{40}\left(\tfrac{1}{j}Z_I^{1}Z_{LO,I}^{1} - \tfrac{1}{j}Z_I^{-1}Z_{LO,I}^{-1} + Z_{I,2}^{1}Z_{LO,I}^{1} + Z_{I,2}^{-1}Z_{LO,I}^{-1} - \tfrac{1}{j}Z_I^{-1}Z_{LO,I}^{1} + \tfrac{1}{j}Z_I^{1}Z_{LO,I}^{-1} + Z_{I,2}^{-1}Z_{LO,I}^{1} + Z_{I,2}^{1}Z_{LO,I}^{-1}\right)$$

$$\tfrac{1}{40}\left(\tfrac{1}{j}\left[j2\sin((\omega_I+\omega_{LO,I})t)\right] + \left[2\cos((\omega_{I,2}+\omega_{LO,I})t)\right] + \tfrac{1}{j}\left[j2\sin((\omega_I-\omega_{LO,I})t)\right] + \left[\cos((\omega_{I,2}-\omega_{LO,I})t)\right]\right)$$

$$\tfrac{1}{20}\left[\sin((2000+4.8\times10^9)\pi t) + \sin((2000-4.8\times10^9)\pi t) + \cos((4000+4.8\times10^9)\pi t) + \cos((4000-4.8\times10^9)\pi t)\right]$$

$X_2 \rightarrow$

$$\tfrac{1}{40}\left(Z_Q^{1}Z_{LO,Q}^{1} + Z_Q^{-1}Z_{LO,Q}^{-1} - \tfrac{1}{j}Z_{Q,2}^{1}Z_{LO,Q}^{1} + \tfrac{1}{j}Z_{Q,2}^{-1}Z_{LO,Q}^{-1} - Z_Q^{1}Z_{LO,Q}^{-1} - Z_Q^{-1}Z_{LO,Q}^{1} + \tfrac{1}{j}Z_{Q,2}^{1}Z_{LO,Q}^{-1} - \tfrac{1}{j}Z_{Q,2}^{-1}Z_{LO,Q}^{1}\right)$$

$$\tfrac{1}{40}\left(\left[2\cos((\omega_Q+\omega_{LO,Q})t)\right] - \tfrac{1}{j}\left[j2\sin((\omega_{Q,2}+\omega_{LO,Q})t)\right] - \left[2\cos((\omega_Q-\omega_{LO,Q})t)\right] + \tfrac{1}{j}\left[j2\sin((\omega_{Q,2}-\omega_{LO,Q})t)\right]\right)$$

$$\tfrac{1}{20}\left[\cos((6000+4.8\times10^9)\pi t) - \cos((6000-4.8\times10^9)\pi t) - \sin((8000+4.8\times10^9)\pi t) + \sin((8000-4.8\times10^9)t)\right]$$

$$V_{RF}(t) = 10(X_1 + X_2)$$

$$\boxed{\begin{aligned} V_{RF}(t) = \tfrac{1}{2}\Big[&\sin((2000+4.8\times10^9)\pi t) + \sin((2000-4.8\times10^9)\pi t) - \sin((8000+4.8\times10^9)\pi t) + \sin((8000-4.8\times10^9)t) \\ &+ \cos((4000+4.8\times10^9)\pi t) + \cos((4000-4.8\times10^9)\pi t) + \cos((6000+4.8\times10^9)\pi t) - \cos((6000-4.8\times10^9)\pi t)\Big] \end{aligned}}$$

b. $F\{V_{RF}(t)\} = \frac{1}{2}\left(\sum F\{\text{each sin and cos in } V_{RF}(t)\}\right)$

$\rightarrow F\{\sin(\omega_1 t)\} = \frac{1}{j2} F\{e^{j\omega t} - e^{-j\omega t}\}$ and $F\{\cos(\omega_1 t)\} = \frac{1}{2} F\{e^{j\omega t} + e^{-j\omega t}\}$

$\rightarrow F\{e^{\pm j\omega t}\} = 2\pi\,\delta(\omega \mp \omega_1)$

$$\boxed{\begin{aligned} F\{V_{RF}(t)\} = \tfrac{1}{2}\Big(2\pi\Big[&\delta(\omega-(2000+4.8\times10^9)\pi) - \delta(\omega+(2000+4.8\times10^9)\pi) + \delta(\omega-(2000-4.8\times10^9)\pi) - \delta(\omega+(2000-4.8\times10^9)\pi) \\ &- \delta(\omega-(8000+4.8\times10^9)\pi) + \delta(\omega+(8000+4.8\times10^9)\pi) + \delta(\omega-(8000-4.8\times10^9)\pi) - \delta(\omega+(8000-4.8\times10^9)\pi) \\ &+ \delta(\omega-(4000+4.9\times10^9)\pi) + \delta(\omega+(4000+4.9\times10^9)\pi) + \delta(\omega-(4000-4.9\times10^9)\pi) + \delta(\omega+(4000-4.9\times10^9)\pi) \\ &+ \delta(\omega-(6000+4.9\times10^9)\pi) + \delta(\omega+(6000+4.9\times10^9)\pi) - \delta(\omega-(6000-4.9\times10^9)\pi) - \delta(\omega+(6000-4.9\times10^9)\pi) \end{aligned}}$$

c. the receiver could use a demodulator to determine the two message signals from the received signal.

(4) $V_{R,I}(t) = \dfrac{V_{RF}(t) \cdot V_{LO,I}(t)}{V_0}$, $V_{LO,I}(t) = 1V \cos(\omega_{LO} t)$

$\omega_{LO} = 2\pi \cdot 2.4\,GHz$

$V_{R,Q}(t) = \dfrac{V_{RF}(t)\, V_{LO,Q}(t)}{V_0}$, $V_{LO,Q}(t) = -1V \sin(\omega_{LO} t)$

$V_{RF}(t) = \dfrac{1}{V_0}\Big(\underbrace{V_I(t)\, V_{LO,I}(t)}_{V_{RF,I}(t)} + \underbrace{V_Q(t)\, V_{LO,Q}(t)}_{V_{RF,Q}(t)}\Big)$. $V_I(t) = (0.1V)\big[\sin$

$$V_{R,I}(t) = \underbrace{\frac{V_{RF,I}(t) \cdot V_{LO,I}(t)}{V_0^2}}_{= V_I(t)\,\frac{(1V)^2}{V_0^2}\cos^2(\omega_{LO} t)} + \underbrace{\frac{V_{RF,Q}(t) \cdot V_{LO,I}(t)}{V_0^2}}_{= \frac{-V_Q(t)}{V_0^2}(1V)^2 \cos(\omega_{LO} t)\sin(\omega_{LO} t)}$$

$V_I(t) = 0.1V$

$$V_{R,I}(t) = \overset{\frac{1}{V_0^2}}{(100V)}\left[\cos^2(\omega_{LO} t) \cdot V_I(t) - V_Q(t)\cos(\omega_{LO} t)\sin(\omega_{LO} t)\right]$$

similarly:

$$V_{R,Q}(t) = (100V)\left[-V_I(t)\cos(\omega_{LO} t)\sin(\omega_{LO} t) + V_Q(t)\sin^2(\omega_{LO} t)\right]$$

NB: $\cos^2(\omega t) = \frac{1}{2}\left[\cos(2\omega t) + 1\right]$

$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta)$

$\sin^2(\theta) = \frac{1}{2}\left[1 - \cos(2\theta)\right]$

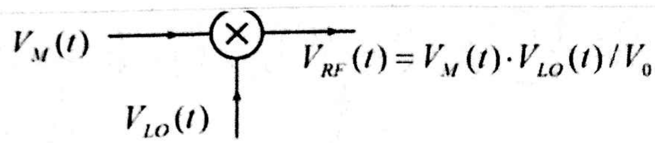
so $\boxed{V_{R,I}(t) = (100V)\cdot\frac{1}{2}\Big[\underbrace{V_I(t)}_{\text{transmitted message}} + \underbrace{V_I(t)\cos(2\omega_{LO}t) - V_Q(t)\sin(2\omega_{LO}t)}_{\text{removed by low-pass filter}}\Big]}$

+

$\boxed{V_{R,Q}(t) = (50V)\Big[\underbrace{V_Q(t)}_{\text{transmitted message}} - \underbrace{V_Q(t)\cos(2\omega_{LO}t) - V_I(t)\sin(2\omega_{LO}t)}_{\text{removed by filter (low-pass)}}\Big]}$

b) the indicated low-pass filters remove the excess signals being modulated at $2\omega_{LO}$ → only want the original transmitted messages $V_I(t)$ + $V_Q(t)$

Problem 5: Returning to problem 1, the local oscillator is again a 1.6MHz cosine wave of 1 V amplitude. The message $V_M(t)$ is a 100 mV *sine* wave at 1kHz, to which we have added a +200 mV DC voltage. V_0 is 0.1 Volts



(a) Compute $V_{RF}(t)$. (b) Make a clean graph of this using your favorite computer software. (c) Compute the Fourier Transform of $V_{RF}(t)$. Note carefully the phases of the sidebands. (d) If $V_{RF}(t)$ is delivered to a 50 Ohm load resistor, compute the *power* at each frequency in its Fourier spectrum. (e) Comment on how much RF power, as a fraction of the total RF power, has been devoted to radiating the 1.6 MHz carrier.

a. $V_{LO}(t)=\cos(3.2\times10^6\pi t)$, $V_M(t)=0.1\sin(2000\pi t)+0.2$, $V_0=0.1V$

$$V_{LO}(t)=\frac{1}{2}e^{j\omega_{LO}t}+\frac{1}{2}e^{-j\omega_{LO}t}=\frac{1}{2}Z_{LO}^{1}+\frac{1}{2}Z_{LO}^{-1}$$

$$V_M(t)=\frac{1}{20j}e^{j\omega_M t}-\frac{1}{20j}e^{-j\omega_M t}+\frac{1}{5}=\frac{1}{20j}Z_M^{1}-\frac{1}{20j}Z_M^{-1}+\frac{1}{5}$$

$$V_M(t)\cdot V_{LO}(t)=\left(\frac{1}{j20}Z_M^{1}-\frac{1}{j20}Z_M^{-1}+\frac{1}{5}\right)\left(\frac{1}{2}Z_{LO}^{1}+\frac{1}{2}Z_{LO}^{-1}\right)=\frac{1}{j40}\left[\left(Z_M^{1}Z_{LO}^{1}-Z_M^{-1}Z_{LO}^{-1}\right)+\left(Z_M^{1}Z_{LO}^{-1}-Z_M^{-1}Z_{LO}^{1}\right)\right]+\frac{1}{10}\left(Z_{LO}^{1}+Z_{LO}^{-1}\right)$$

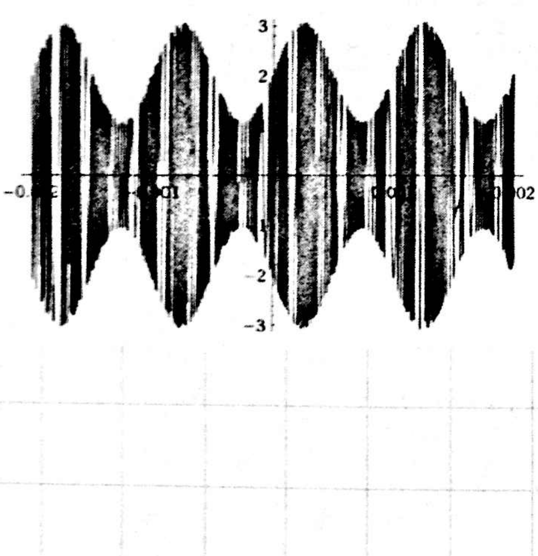
$$=\frac{1}{j40}\left[\left(e^{j(\omega_M+\omega_{LO})t}-e^{-j(\omega_M+\omega_{LO})t}\right)+\left(e^{j(\omega_M-\omega_{LO})t}-e^{-j(\omega_M-\omega_{LO})t}\right)\right]+\frac{1}{10}\left(e^{j\omega_{LO}t}+e^{-j\omega_{LO}t}\right)$$

$$=\frac{1}{j40}\left[j2\sin((\omega_M+\omega_{LO})t)+j2\sin((\omega_M-\omega_{LO})t)\right]+\frac{1}{10}\left(2\cos(\omega_{LO}t)\right)$$

$$V_{RF}(t)=V_M(t)\cdot V_{LO}(t)/V_0=\boxed{\frac{1}{2}\left[\sin(3.202\times10^6\pi t)-\sin(3.198\times10^6\pi t)+4\cos(3.2\times10^6\pi t)\right]}$$

b. plot $0.5\,(4\cos(1.00531\times10^7\,t) - \sin(1.00468\times10^7\,t)+\sin(1.00594\times10^7\,t))$ $\quad t = -\frac{1}{500}$ to $\frac{1}{500}$

Plot



c. $F\{V_{RF}(t)\} = \frac{1}{2}\Big[F\{\sin(\underbrace{3.202\times10^6\pi}_{\omega_1} t)\} - F\{\sin(\underbrace{3.198\times10^6\pi}_{\omega_2} t)\} + 4F\{\cos(\underbrace{3.2\times10^6\pi}_{\omega_3} t)\}\Big]$

$$= \frac{1}{2}\left[\frac{1}{j2}\left(F\{e^{j\omega_1 t}\} - F\{e^{-j\omega_1 t}\}\right) - \frac{1}{j2}\left(F\{e^{j\omega_2 t}\} - F\{e^{-j\omega_2 t}\}\right) + 4\left(F\{e^{j\omega_3 t}\} + F\{e^{-j\omega_3 t}\}\right)\right]$$

$$= \frac{1}{j4}\left[2\pi\left(\delta(\omega-\omega_1) - \delta(\omega+\omega_1) - \delta(\omega-\omega_2) + \delta(\omega+\omega_2) + 2\delta(\omega-\omega_3) + 2\delta(\omega+\omega_3)\right]\right.$$

$$= -j\pi/2\,\delta(\omega-3.202\times10^6\pi) + j\pi/2\,\delta(\omega+3.202\times10^6\pi) + j\pi/2\,\delta(\omega-3.198\times10^6\pi) - j\pi/2(\omega+3.198\times10^6\pi) - j\pi\,\delta(\omega-3.2\times10^6\pi) - j\pi\delta(\omega+3.2\times10^6\pi)$$

$$\boxed{F\{V_{RF}(t)\} = j\frac{\pi}{2}\left[\delta(\omega+3.202\times10^6\pi) - \delta(\omega-3.202\times10^6\pi) + \delta(\omega-3.198\times10^6\pi) - (\omega+3.198\times10^6\pi) - 2\delta(\omega-3.2\times10^6\pi) - 2\delta(\omega+3.2\times10^6\pi)\right]}$$

d. $P = V^2/R$

$$P_1 = \left(j\frac{\pi}{2}\delta(\omega+3.202\times10^6\pi)\right)^2 \cdot \frac{1}{50\Omega} = -\frac{\pi^2}{400}\delta(\omega+3.202\times10^6\pi)^2$$

$$P_2 = -\left(j\frac{\pi}{2}\delta(\omega-3.202\times10^6\pi)\right)^2 \cdot \frac{1}{50\Omega} = +\frac{\pi^2}{400}\delta(\omega-3.202\times10^6\pi)^2$$

$$P_3 = \left(j\frac{\pi}{2}\delta(\omega+3.198\times10^6\pi)\right)^2 \cdot \frac{1}{50\Omega} = -\frac{\pi^2}{400}\delta(\omega+3.198\times10^6\pi)^2$$

$$P_4 = -\left(j\frac{\pi}{2}\delta(\omega-3.198\times10^6\pi)\right)^2 \cdot \frac{1}{50\Omega} = +\frac{\pi^2}{400}\delta(\omega-3.198\times10^6\pi)^2$$

$$P_5 = -\left(j\pi\,\delta(\omega-3.2\times10^6\pi)\right)^2 \cdot \frac{1}{50\Omega} = +\frac{\pi^2}{50}\delta(\omega-3.2\times10^6\pi)^2$$

$$P_6 = -\left(j\pi\,\delta(\omega+3.2\times10^6\pi)\right)^2 \cdot \frac{1}{50\Omega} = +\frac{\pi^2}{50}\delta(\omega+3.2\times10^6\pi)^2$$

e. $1/3$ of the RF power was devoted to radiating the 1.6 MHz carrier